Junior problems

J547. Find all primes p such that

$$\frac{2^{p+2}-1}{p}$$

is prime.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by Ashley Simone, SUNY Brockport

If

$$\frac{2^{p+2}-1}{p}$$

is a prime number then

$$p|2^{p+2}-1$$

In particular, this implies that p is an odd prime. Then, by Fermat's Theorem,

$$p|2^{p-1}-1$$

which implies that

$$p|(2^{p+2}-1)-(2^{p-1}-1)=2^{p+2}-2^{p-1}=7\cdot 2^{p-1}$$

Since p is an odd prime, we conclude that $p=7$. Therefore the only possible solution is $p=7$. In this case

$$\frac{2^{p+2}-1}{p}=\frac{2^9-1}{7}=73$$

which is a prime number.

Thus 7 is the only solution of this problem.

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Sebastian Fernandez, Costa Rican Olympiad Team; Archisman Nandy, St.Agnes School-Kharagpur, India; Corneliu Mănescu-Avram, Ploieşti, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Henry Ricardo, Westchester Area Math Circle; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Joel Schlosberg, Bayside, NY, USA; David E. Manes, Oneonta, NY, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Polyahedra, Polk State College, USA; Fred Frederickson, Utah Valley University, UT, USA; Anderson Torres, Brazil; Lazar Ilic; Grant Blitz, Glenview, IL, USA.

J548. Let a, b, c, x, y be positive real numbers such that $x + y = 1$. Prove that

$$\sqrt{\frac{a^3}{xa+yb}}+\sqrt{\frac{b^3}{xb+yc}}+\sqrt{\frac{c^3}{xc+ya}} \geq a+b+c.$$

Proposed by Mircea Becheanu, Canada

First solution by Henry Ricardo, Westchester Area Math Circle

We use the Cauchy-Schwarz inequality to see that

$$\begin{aligned}\sum_{cyc}\sqrt{\frac{a^3}{xa+yb}} &= \sum_{cyc}\sqrt{\frac{a^4}{a(xa+yb)}} = \sum_{cyc}\frac{a^2}{\sqrt{a(xa+yb)}}\\ &\geq \frac{(a+b+c)^2}{\sum_{cyc}\sqrt{a(xa+yb)}} \geq a+b+c \iff \sum_{cyc}\sqrt{a(xa+yb)} \leq a+b+c.\end{aligned}$$

Applying the Cauchy-Schwarz inequality again, we have

$$\sum_{cyc}\sqrt{a(xa+yb)} \leq \sqrt{a+b+c}\cdot\sqrt{(x+y)(a+b+c)} = a+b+c,$$

and we are finished. Equality holds if and only if $a = b = c$.

Second solution by Polyahedra, Polk State College, USA

Applying Jensen's inequality to the convex function $1/\sqrt{t}$, we get

$$\begin{aligned}&\frac{a}{a+b+c}\cdot\frac{1}{\sqrt{x+yb/a}}+\frac{b}{a+b+c}\cdot\frac{1}{\sqrt{x+yc/b}}+\frac{c}{a+b+c}\cdot\frac{1}{\sqrt{x+ya/c}}\\ &\geq \frac{1}{\sqrt{\frac{ax+yb+bx+yc+cx+ya}{a+b+c}}} = 1.\end{aligned}$$

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Ace Kim, Northern Valley Regional High School at Old Tappan, NJ, USA; Corneliu Mănescu-Avram, Ploiești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Daniel Văcaru, Pitești, Romania; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J549. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} - \frac{9}{a+b+c} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

First solution by Henry Ricardo, Westchester Area Math Circle

We use the result that $x/y^2 + y/x^2 \ge 1/x + 1/y$ for $x, y > 0$:

$$\frac{x}{y^2} + \frac{y}{x^2} \ge \frac{1}{x} + \frac{1}{y} \iff \frac{x^3+y^3}{x+y} \ge xy \iff x^2 - xy + y^2 \ge xy \iff (x-y)^2 \ge 0.$$

Equality holds if and only if $x = y$.

Now

$$\begin{aligned} \frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} &= \left(\frac{a}{c^2} + \frac{c}{a^2}\right) + \left(\frac{b}{c^2} + \frac{c}{b^2}\right) + \left(\frac{a}{b^2} + \frac{b}{a^2}\right) \\ &\ge \left(\frac{1}{c} + \frac{1}{a}\right) + \left(\frac{1}{c} + \frac{1}{b}\right) + \left(\frac{1}{b} + \frac{1}{a}\right) = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \end{aligned} \tag{1}$$

and

$$\frac{9}{a+b+c} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad \text{or} \quad -\frac{9}{a+b+c} \ge -\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \tag{2}$$

by the Harmonic Mean-Arithmetic Mean inequality.

Combining (1) and (2), we see that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} - \frac{9}{a+b+c} \ge 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Equality holds if and only if $a = b = c$.

Second solution by Henry Ricardo, Westchester Area Math Circle

The homogeneity of the inequality allows us to assume $a+b+c=1$. Then the inequality becomes

$$\frac{1-c}{c^2} + \frac{1-a}{a^2} + \frac{1-b}{b^2} - 9 \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

The Harmonic Mean–Arithmetic Mean inequality gives us $(a+b+c)(1/a+1/b+1/c) \geq 9$, or $-9 \geq -(1/a + 1/b + 1/c)$, so that our inequality becomes

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Now

$$\begin{aligned}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \frac{1}{2}\sum_{cyclic}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) &\geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \\ &\geq 3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \iff a+b+c \geq 3(ab+bc+ca).\end{aligned}$$

Since Maclaurin's inequality gives us $(a+b+c)/3 \geq \sqrt{(ab+bc+ca)/3}$, we have $1/3 \geq \sqrt{(ab+bc+ca)/3}$, or $1 \geq 3(ab+bc+ca)$, and we are done. Equality holds if and only if $a=b=c$.

Also solved by Titu Zvonaru, Comănești, Romania; Polyahedra, Polk State College, USA; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Fred Frederickson, Utah Valley University, UT, USA; Costa Rican Olympiad Team; Anderson Torres, Brazil; Alejandro Campos, Costa Rican Olympiad Team; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Daniel Văcaru, Pitești, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

J550. Let a, b, c be real numbers with $a, b \leq c$, such that $abc = 1$ and $ab + bc + ca = 0$. Find the greatest real number k such that

$$|a + b| \geq k|c|.$$

Proposed by Ayashi Jain, Gurgaon, Haryana, India

Solution by Polyahedra, Polk State College, USA

Suppose a, b, c satisfy the conditions. if $c < 0$, then one of a, b is positive, thus greater than c. So $c > 0$. Since $ab = 1/c$ and $a + b = -1/c^2$

$$1/c^4 = (a + b)^2 \geq 4ab = 4/c.$$

Thus, $|a + b|/|c| = 1/c^3 \geq 4$. Equality holds if $a = b = -\sqrt[3]{2}$ and $c = 1/\sqrt[3]{4}$. Therefore, the greatest k is 4.

Also solved by Titu Zvonaru, Comănești, Romania; Fred Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Daniel Văcaru, Pitești, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Todor Zaharinov, Sofia, Bulgaria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

J551. Let $ABCD$ be a square and let M be a point on side CD. The lines AM and BD intersect in E. The perpendicular in E on AM intersects BC in N, and AN intersects BD in F. Let K be the intersection point of EN and FM. Prove that AK is perpendicular to MN.

Proposed by Mircea Becheanu, Canada

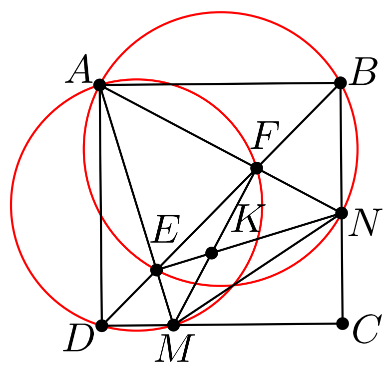
First solution by the author

With the notations from the statement, we prove the following

Lemma: Let $ABCD$ be a square and M be an interior point on the side DC. The lines AM and BD intersect in E. The perpendicular in E on AM intersects BC in N. Then $\angle MAN = 45°$.

Proof: The quadrilateral $AENB$ is cyclic because $\angle NEA = \angle NBA = 90°$. Then $\angle EAN = \angle EBN = 45°$. Back to the solution, we draw the perpendicular line from F to AN, to intersect DC in a point M', such that $\angle NAM' = 45°$. This shows that $M \equiv M'$, hence $MF \perp AN$. Consider the triangle MAN. The point K is its orthocenter, hence $AK \perp MN$.

Second solution by Polyahedra, Polk State College, USA



By construction, A, E, N, B lie on the circle with diameter AN, so $\angle MAN = \angle EBN = 45° = \angle MDF$. Therefore, A, D, M, F lie on the circle with diameter AM. Hence, $MF \perp AN$, that is, K is the orthocenter of $\triangle AMN$, completing the proof.

Also solved by Titu Zvonaru, Comănești, Romania; Chistopher Lee, Singapore American School, Singapore; Fred Frederickson, Utah Valley University, UT, USA; Anderson Torres, Brazil; Grant Blitz, Glenview, IL, USA; Ernesto Delgado, Kristel Acuna, Leonardo Loria, Maricruz Vasquez, Alejandro Campos, Costa Rican Olympiad Team; Corneliu Mănescu-Avram, Ploieşti, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

J552. Let x, y, z be positive real numbers with $xy + yz + zx + xyz = 4$. Prove that

$$2\left(\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1}\right) \le 3\sqrt{(x+1)(y+1)(z+1)}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

First solution by Corneliu Mănescu-Avram, Ploieşti, Romania

Denote $x+1 = a^2, y+1 = b^2, z+1 = c^2$, where a, b, c are real numbers greater than 1. Then we have to prove $a^2b^2c^2 = a^2 + b^2 + c^2 + 2$ implies $2(a+b+c) \le 3abc$, that is

$$c^2 = \frac{a^2 + b^2 + c^2}{a^2b^2 - 1} \text{ implies } c \ge \frac{2(a+b)}{3ab-2}.$$

For $s = a + b, p = ab$, we have to prove that

$$\frac{s^2 - 2p + 2}{p^2 - 1} \ge \left(\frac{2s}{3p-2}\right)^2,$$

which is equivalent to

$$s^2 \ge \frac{2(p-1)(3p-2)^2}{5p^2 - 12p + 8}.$$

Since $s^2 \ge 4p$, it suffices to prove that

$$4p \ge \frac{2(p-1)(3p-2)^2}{5p^2 - 12p + 8},$$

equivalent to $(p+1)(p-2)^2 \ge 0$, which is true. Equiality holds only for $p = 2$, that is, only for $x = y = z = 1$.

Second solution by Polyahedra, Polk State College, USA
Let $a = \frac{1}{x+2}$, $b = \frac{1}{y+2}$, and $c = \frac{1}{z+2}$.Then

$$a + b + c = \frac{xy + yz + zx + 4(x + y + z) + 12}{xyz + 2(xy + yz + zx) + 4(x + y + z) + 8} = 1,$$

so $x+1 = \frac{1-a}{a} = \frac{b+c}{a}$, $y+1 = \frac{c+a}{b}$, and $z+1 = \frac{a+b}{c}$. It is well known and easy to prove that $(a+b+c)(ab+bc+ca) \le \frac{9}{8}(a+b)(b+c)(c+a)$. Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned} 2\left(\sqrt{(b+c)bc} + \sqrt{(c+a)ca} + \sqrt{(a+b)ab}\right) &\le 2\sqrt{2(b+c+a)(bc+ca+ab)} \\ &\le 3\sqrt{(b+c)(c+a)(a+b)}. \end{aligned}$$

Dividing both sides by $\sqrt{abc}$ completes the proof.

Also solved by Titu Zvonaru, Comănești, Romania; Fred Frederickson, Utah Valley University, UT, USA; Anderson Torres, Brazil; Lazar Ilic; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

Senior problems

S547. Let a and b be positive real numbers less than 2 such that $ab = 2$. Solve in real numbers the equation

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 = 9.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Solution by the author

We have

$$x^2 + ax + b = \left(x + \frac{a}{2}\right)^2 + \frac{-\Delta_1}{4},$$

where $\dfrac{-\Delta_1}{4} = \dfrac{4b - a^2}{4} = \dfrac{8 - a^3}{4a} > 0$.

Similarly,

$$x^2 + bx + a = \left(x + \frac{b}{2}\right)^2 + \frac{-\Delta_2}{4},$$

where $\dfrac{-\Delta_2}{4} = \dfrac{4a - b^2}{4} = \dfrac{8 - b^3}{4b} > 0$. Then

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 \geq \frac{4(8 - a^3)}{4a} \cdot \frac{8 - b^3}{4b} = \frac{64 - 8a^3 - 8b^3 + 8}{8} + a^3 + b^3 = 9,$$

with equality if and only if $x + \dfrac{a}{2} = x + \dfrac{b}{2} = 0$.

Hence the equation is solvable in real numbers if and only if $a = b = \sqrt{2}$, in which case the unique solution is $x = -\dfrac{\sqrt{2}}{2}$.

Also solved by Titu Zvonaru, Comănești, Romania; Dao Quang Anh, Everest School, Hoang Quoc Viet, Ha Noi, Vietnam; Marie-Nicole Gras, Le Bourg d'Oisans, France.

S548. Let a, b, c, d be nonnegative real numbers such that $a+b+c+d=10$. Prove that

$$6a+2ab+abc+abcd \le 96.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let

$$f(a,b,c,d)=6a+2ab+abc+abcd \quad \text{and} \quad g(a,b,c,d)=a+b+c+d-10.$$

The method of Lagrange multipliers yields the equations

$$\lambda = abc \tag{1}$$

$$\lambda = ab+abd \tag{2}$$

$$\lambda = 2a+ac+acd \tag{3}$$

$$\lambda = 6+2b+bc+bcd \tag{4}$$

Because b, c, and d are nonnegative, $\lambda \neq 0$ by equation (4). This then implies that a, b, and c are not equal to 0. Now, multiply equation (2) by c and combine with equation (1) to obtain $d=c-1$. Next, substitute $d=c-1$ and $\lambda=abc$ into equation (3) and solve for

$$b=c+\frac{2}{c}.$$

To determine a in terms of c, substitute $d=c-1$, $\lambda=abc$, and $b=c+\frac{2}{c}$ into equation (4):

$$a=\frac{6}{c^2+2}+c+\frac{2}{c}.$$

The requirement that $a+b+c+d=10$ then becomes

$$\frac{6}{c^2+2}+2\frac{c^2+2}{c}+2c-1=10$$

or

$$(c-2)(4c^3-3c^2+6c-4)=0.$$

For d to be greater than or equal to 0, c must be greater than or equal to 1. With $c>1$, $4c^3-3c^2+6c-4>0$, so c must be 2. Then $a=4$, $b=3$, and $d=1$. The maximum value of $f(a,b,c,d)$ is then

$$f(4,3,2,1)=6(4)+2(4)(3)+4(3)(2)+4(3)(2)(1)=96;$$

that is,

$$6a+2ab+abc+abcd \le 96.$$

Also solved by Titu Zvonaru, Comănești, Romania; Marie-Nicole Gras, Le Bourg d'Oisans, France; Spyros Kallias, Volos, Greece.

S549. Let a, b, c be positive real numbers such that $a+b+c+abc=4$. Prove that

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \le \sqrt{1+4a-a^2}+\sqrt{1+4b-b^2}+\sqrt{1+4c-c^2} \le ab+bc+ca+3$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA
First, note that in fact holds inequality

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}+3 \le \sqrt{1+4a-a^2}+\sqrt{1+4b-b^2}+\sqrt{1+4c-c^2} \le ab+bc+ca+3. \tag{1}$$

Indeed, since $2\sqrt{bc} \le b+c = 4-a-abc \iff 2a\sqrt{bc} \le 4a-a^2-a^2bc \iff$

$$1+2a\sqrt{bc}+a^2bc \le 1+4a-a^2 \iff 1+a\sqrt{bc} \le \sqrt{1+4a-a^2}$$

and

$$a\left(\frac{b+c}{2}\right)^2 \ge abc = 4-a-b-c = 4-a-\frac{2(b+c)}{2} \iff a\left(\frac{b+c}{2}\right)^2+\frac{2(b+c)}{2} \ge 4-a \iff$$

$$a^2\left(\frac{b+c}{2}\right)^2+2a\cdot\frac{b+c}{2} \ge 4a-a^2 \iff \left(\frac{a(b+c)}{2}+1\right)^2 \ge 4a-a^2+1 \iff$$

$$\frac{a(b+c)}{2}+1 \ge \sqrt{1+4a-a^2}$$

then $1+a\sqrt{bc} \le \sqrt{1+4a-a^2} \le \dfrac{a(b+c)}{2}+1$ and, therefore,

$$\sum_{cyc}\left(1+a\sqrt{bc}\right) \le \sum_{cyc}\sqrt{1+4a-a^2} \le \sum_{cyc}\left(\frac{a(b+c)}{2}+1\right) \iff (1).$$

Equalities in (1) occurs iff $a=b=c=1$.

Also solved by Titu Zvonaru, Comănești, Romania; Arighna Pan, Nabadwip Vidyasagar College, India; Prodromos Fotiadis, Nikiforos High School, Drama, Greece.

S550. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2+2ab}+\sqrt{b^2+2bc}+\sqrt{c^2+2ca} \geq \sqrt{3ab}+\sqrt{3bc}+\sqrt{3ca}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author

Because $u^2+v^2+w^2 \geq \dfrac{(u+v+w)^2}{3}$ and $u^2+v^2+w^2 \geq uv+vw+wu$, we have

$$\sum_{cyc}\sqrt{a^2+2ab} \geq \sum_{cyc}\frac{a+\sqrt{ab}+\sqrt{ab}}{\sqrt{3}} = \frac{\sqrt{3}}{3}\cdot\sum_{cyc}a+\frac{2\sqrt{3}}{3}\cdot\sum_{cyc}\sqrt{ab} \geq \frac{3\sqrt{3}}{3}\cdot\sum_{cyc}\sqrt{ab} = \sum_{cyc}\sqrt{3ab},$$

as desired.

Also solved by Titu Zvonaru, Comănești, Romania; Arkady Alt, San Jose, CA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Marie-Nicole Gras, Le Bourg d'Oisans, France.

S551. Let a, b, c be the side lengths of a triangle with inradius r and circumradius R. Prove that

$$\frac{R}{r} + (1+\sqrt{5}) \geq (3+\sqrt{5}) \cdot \frac{a^2+b^2+c^2}{ab+bc+ca}$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

Solution by the author

Without loss of generality, we may assume that $c = \min\{a,b,c\}$. Using the Ravi's substitutions i.e. $a = y+z$, $b = z+x$, $c = x+y$, $x, y, z > 0$, and the basic triangle properties, the inequality can be rewritten as follows

$$\frac{(x+y)(y+z)(z+x)}{4xyz} \geq 2(3+\sqrt{5})\frac{x^2+y^2+z^2+xy+yz+zx}{x^2+y^2+z^2+3(xy+yz+zx)} - 1 - \sqrt{5},$$

or

$$\frac{(x+y)(y+z)(z+x) - 8xyz}{4xyz} \geq (3+\sqrt{5})\frac{x^2+y^2+z^2-xy-yz-zx}{x^2+y^2+z^2+3(xy+yz+zx)},$$

that is

$$\frac{2z(x-y)^2 + (x+y)(x-z)(y-z)}{4xyz} \geq (3+\sqrt{5})\frac{(x-y)^2+(x-z)(y-z)}{x^2+y^2+z^2+3(xy+yz+zx)}.$$

Since $c = \min\{a,b,c\}$ it follows that $z = \max\{x,y,z\}$ and from here, we have

$$\begin{aligned} x^2+y^2+z^2+3(xy+yz+zx) &\geq 2xy + xy + 3xy + 6z\sqrt{xy} \\ &\geq 12xy \geq 2(3+\sqrt{5})xy. \end{aligned}$$

It remains to show that

$$(x+y)\left(x^2+y^2+z^2+3xy+3yz+3zx\right) \geq 4(3+\sqrt{5})xyz,$$

that is

$$x^3+y^3+4xy(x+y)+(x+y)z^2+3z(x+y)^2 \geq 12xyz + 4\sqrt{5}xyz.$$

Using the AM-GM Inequality, it suffices to show that

$$2xy\sqrt{xy} + 8xy\sqrt{xy} + 2\sqrt{xy}z^2 + 3z(x-y)^2 \geq 4\sqrt{5}xyz,$$

that is

$$2\sqrt{xy}\left(z - \sqrt{5xy}\right)^2 + 3z(x-y)^2 \geq 0,$$

clearly true.

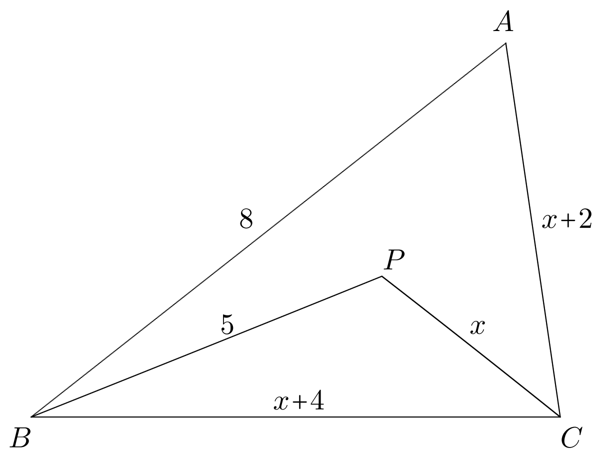
The equality holds when $x = y = z$ which means $a = b = c$ or when $x = y$, $z = \sqrt{5xy}$ which means $a = b = \dfrac{2c}{\sqrt{5}-1}$ (or any cyclic permutation). In other words, the equality holds for the equilateral triangle, respectively the isosceles triangle in which $A = B = 72°$, $C = 36°$.

Also solved by Titu Zvonaru, Comănești, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Arkady Alt, San Jose, CA, USA; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Marie-Nicole Gras, Le Bourg d'Oisans, France.

S552. Find all triangles ABC with $AB = 8$ for which there is an interior point P such that $PB = 5$, PC, AC, BC is an arithmetic sequence with common difference 2 and $\angle BPC = 2\angle BAC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Marie-Nicole Gras, Le Bourg d'Oisans, France



Put $x = PC$, then $AC = x + 2$ and $BC = x + 4$; since $BC + AC > AB$, we deduce $x > 1$.

In $\triangle ABC$, we obtain, by the cosine relation

$$\cos(\angle BAC) = \frac{AB^2 + AC^2 - BC^2}{2AB \cdot AC} = \frac{64 + (x+2)^2 - (x+4)^2}{16(x+2)} = \frac{13 - x}{4(x+2)}.$$

In $\triangle PBC$, we obtain, in the same manner

$$\cos(\angle BPC) = \frac{PB^2 + PC^2 - BC^2}{2PB \cdot PC} = \frac{25 + x^2 - (x+4)^2}{10x} = \frac{9 - 8x}{10x}.$$

Since, by assumption, we have $\angle BPC = 2\angle BAC$, we deduce from the relation $\cos(\angle BPC) = 2\cos^2(\angle BAC) - 1$, that x is a solution of the equation

$$\frac{9 - 8x}{10x} = 2\left(\frac{13 - x}{4(x+2)}\right)^2 - 1.$$

After cleaning the denominators, we obtain that x is a solution of the equation

$$x^3 + 66x^2 - 223x + 48 = (x - 3)(x^2 + 69x - 16) = 0.$$

Polynomial $x^2 + 69x - 16$ has 2 roots, $x_1 < 0$ and $x_2 < 1$; there are not suitable. It follows that the unique solution is

$$x = 3, \quad AB = 8,\ BC = 7,\ CA = 5.$$

We note that $\cos(\angle BAC) = \dfrac{64 + 25 - 49}{80} = \dfrac{1}{2}$, whence $\angle BAC = 60^\circ$.

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Fred Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefallonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

Undergraduate problems

U547. Let a, b, c, d be real numbers such that all solutions of the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + 1022 = 0$$

are real numbers less than -1. Pove that $a + c < b + d$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by the author
We have $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + 1022 = (x - x_1)....(x - x_5)$ and $x_1....x_5 = -1022$. Moreover, because all roots are less than -1 we have $P(-1) > 0$. We consider the inequalities $x_k^2 \geq 4(-1 - x_k)$ for $k = 1, 2, ..., 5$. By multiplication we have:

$$x_1^2 \ldots x_5^2 \geq 4^5(-1 - x_1)\ldots(-1 - x_5) = 4^5 P(-1) = 4^5(a - b + c - d + 1021) =$$

$$= 4^5(a - b + c - d) + 4^5 \cdot 1021$$

Using the equality $x_1....x_5 = -1022$ the conclusion follows.

U548. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan^n x}$$

where n is a positive integer.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Henry Ricardo, Westchester Area Math Circle

Using the identity $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$, we have

$$I(n) = \int_0^{\pi/2} \frac{dx}{1+\tan^n x} = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\,dx = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx.$$

Adding the last two integrals, we see that $2I(n) = \int_0^{\pi/2} 1\,dx = \pi/2$, which yields $I(n) = \pi/4$ for any nonnegative integer n.

Second solution by Henry Ricardo, Westchester Area Math Circle

Denoting the given integral by $I(n)$, the substitution $x \mapsto \arctan t$ gives us

$$I(n) = \int_0^{\infty} \frac{1}{1+t^n} \cdot \frac{1}{1+t^2}\,dt.$$

Then the substitution $t \mapsto 1/t$ yields

$$I(n) = \int_0^{\infty} \frac{1}{1+t^{-n}} \cdot \frac{1}{1+t^{-2}} \cdot \frac{1}{t^2}\,dt = \int_0^{\infty} \frac{1}{1+t^{-n}} \cdot \frac{1}{1+t^2}\,dt.$$

Therefore,

$$2I(n) = \int_0^{\infty} \frac{1}{t^2+1}\left(\frac{1}{1+t^n} + \frac{1}{1+t^{-n}}\right)dt = \int_0^{\infty} \frac{dt}{t^2+1} = \frac{\pi}{2},$$

which implies that $I(n) = \pi/4$ for every nonnegative integer n.

Also solved by Taes Padhihary, Disha Delphi Public School, India; Brian Bradie, Christopher Newport University, Newport News, VA, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Moubinool Omarjee, Paris, France; Olimjon Jalilov, Tashkent, Uzbekistan; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA.

U549. Evaluate

$$\sum_{n=1}^{\infty} \frac{4n-1}{n^2(2n-1)^2}.$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

By partial fractions,

$$\frac{4n-1}{n^2(2n-1)^2} = \frac{4}{(2n-1)^2} - \frac{1}{n^2}.$$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{4n-1}{n^2(2n-1)^2} = 4\left(\frac{\pi^2}{8}\right) - \frac{\pi^2}{6} = \frac{\pi^2}{3}.$$

Also solved by Titu Zvonaru, Comănești, Romania; Taes Padhihary, Disha Delphi Public School, India; Ivko Dimitrić, Pennsylvania State University Fayette, PA, USA; Fred Frederickson, Utah Valley University, UT, USA; Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefallonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Lazar Ilic; Daniel Văcaru, Pitești, Romania; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Prodromos Fotiadis, Nikiforos High School, Drama, Greece; Henry Ricardo, Westchester Area Math Circle; Le Hoang Bao, Tien Giang, Vietnam; Maiteyo Bhattacharjee, IACS, Kolkata, India; Olimjon Jalilov, Tashkent, Uzbekistan; Arkady Alt, San Jose, CA, USA.

U550. Let

$$f_n(x) = (x^2 - x + 1)(x^4 - x^2 + 1)(x^8 - x^4 + 1)\cdots(x^{2^n} - x^{2^{n-1}} + 1)$$

Prove that for $|x| < 1$

$$\frac{1}{3} < \lim_{n\to\infty} f_n(x) \le \frac{4}{3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Brian Bradie, Christopher Newport University, Newport News, VA, USA

Let

$$f_n(x) = \prod_{j=1}^{n}\left(x^{2^j} - x^{2^{j-1}} + 1\right).$$

Then

$$\begin{aligned}(x^2+x+1)f_n(x) &= (x^2+x+1)(x^2-x+1)\prod_{j=2}^{n}\left(x^{2^j} - x^{2^{j-1}} + 1\right)\\ &= (x^4+x^2+1)(x^4-x^2+1)\prod_{j=3}^{n}\left(x^{2^j} - x^{2^{j-1}} + 1\right)\\ &= (x^8+x^4+1)(x^8-x^4+1)\prod_{j=4}^{n}\left(x^{2^j} - x^{2^{j-1}} + 1\right)\\ &= \cdots = \left(x^{2^n} + x^{2^{n-1}} + 1\right)\left(x^{2^n} - x^{2^{n-1}} + 1\right)\\ &= x^{2^{n+1}} + x^{2^n} + 1.\end{aligned}$$

For $|x| < 1$,

$$(x^2+x+1)\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(x^{2^{n+1}} + x^{2^n} + 1\right) = 1.$$

Let

$$g(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{x^2+x+1}.$$

Then

$$g'(x) = -\frac{2x+1}{(x^2+x+1)^2},$$

so g is increasing for $-1 < x < -1/2$ and is decreasing for $-1/2 < x < 1$. Moreover,

$$\lim_{x\to -1^+} g(x) = 1, \quad g\left(-\frac{1}{2}\right) = \frac{4}{3}, \quad \text{and} \quad \lim_{x\to 1^-} g(x) = \frac{1}{3}.$$

Thus, for $|x| < 1$,

$$\frac{1}{3} < \lim_{n\to\infty} f_n(x) \le \frac{4}{3}.$$

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefallonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Henry Ricardo, Westchester Area Math Circle; Mihai Craciun, Mihail Sadoveanu National College, Pașcani, Romania; Olimjon Jalilov, Tashkent, Uzbekistan; Arkady Alt, San Jose, CA, USA.

U551. Let $P(x) = a_0 + a_1 x + \cdots + a_d x^d$ be a polynomial with positive coefficients such that $a_k^2 > 9a_{k-1}a_{k+1}$, for all $k = 1, ..., d-1$. Prove that $P(x)$ has d distinct real roots.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Solution by the author

Let $b_k = 3^{k^2} a_k$. Then $b_{k-1}b_{k+1} = 3^{2k^2+2}a_{k-1}a_{k+1} < 3^{2k^2}a_k^2 = b_k^2$. That is

$$\frac{b_{k-1}}{b_k} < \frac{b_k}{b_{k+1}}.$$

Now, consider the interval $\left(\frac{3^{2m}b_{m-1}}{b_m}, \frac{3^{2m}b_m}{b_{m+1}}\right)$. Note that $\frac{3^{2m}b_{m-1}}{b_m} = \frac{3a_{m-1}}{a_m}$ and $\frac{3^{2m}b_m}{b_{m+1}} = \frac{a_m}{3a_{m+1}}$.

We prove that for these intervals $P(-x)$ is non-zero and has a sign of $(-1)^m$. That is

$$\frac{P(-x)}{(-x)^m} = \cdots + (-a_{m-3}x^{-3} + a_{m-2}x^{-2}) + (-a_{m-1}x^{-1} + a_m - a_{m+1}x) + (a_{m+2}x^2 - a_{m+3}x^3) + \cdots$$

Note that $-a_{m-1}x^{-1} + a_m - a_{m+1}x > a_m\left(-\frac{1}{3} + 1 - \frac{1}{3}\right) > 0$. Further,

$$(-a_{m-2s-1}x^{-1-2s} + a_{m-2s}x^{-2s}) = a_{m-2s}x^{-1-2s}\left(x - \frac{a_{m-2s-1}}{a_{m-2s}}\right).$$

For $x \in \left(\frac{3^{2m}b_{m-1}}{b_m}, \frac{3^{2m}b_m}{b_{m+1}}\right)$, $x - \frac{a_{m-2s-1}}{a_{m-2s}}$ is greater than $\frac{3^{2m}b_{m-1}}{b_m} - \frac{3^{2m-4s-1}b_{m-2s-1}}{b_{m-2s}} > 0$.

By the same argument $a_{m+2s}x^{2s} - a_{m+2s+1}x^{2s+1} = a_{m+2s+1}x^{2s}\left(\frac{a_{m+2s}}{a_{m+2s+1}} - x\right)$.

Now, for $x \in \left(\frac{3^{2m}b_{m-1}}{b_m}, \frac{3^{2m}b_m}{b_{m+1}}\right)$ it follows that

$$\frac{a_{m+2s}}{a_{m+2s+1}} - x > \frac{a_{m+2s}}{a_{m+2s+1}} - \frac{3^{2m}b_m}{b_{m+1}} = 3^{2m+4s+1}\frac{b_{m+2s}}{b_{m+2s+1}} - \frac{3^{2m}b_m}{b_{m+1}} > 0.$$

Hence, in each interval $\left(-\infty, -\frac{a_{d-1}}{3a_d}\right), \left[-\frac{3a_{d-2}}{a_{d-1}}, \frac{a_{d-2}}{3a_{d-1}}\right], \cdots, \left(-\frac{3a_0}{a_1}, 0\right)$ we have one real root and we are done.

U552. Find all polynomials $P(x)$ with real coefficients for which

$$P(P(a+b)) - 2ab(2P(a+b) - ab) \geq P(a^2) + P(b^2) \geq P(a^2+b^2) - P(\sqrt{2}ab)$$

Proposed by Karthik Vedula, James S. Rickards High School, Tallahassee, FL, USA

Solution by the author

$$P(x) \equiv 0, cx^2 \ (c \in [-1,0) \cup \{1\})$$

Plugging in $b = 0$ gives

$$P(P(a)) \geq P(a^2) + P(0) \geq P(a^2) - P(0) \implies P(P(0)) \geq 2P(0) \geq 0$$

Setting $b = -a$ gives

$$P(P(0)) - 4P(0)a^2 + 2a^4 \geq 2P(a^2)$$

Now, we do casework on the small degrees:

1. $\deg P = 0 \implies P(x) = c$. Substituting this in $P(P(0)) \geq 2P(0) \geq 0 \implies c \geq 2c \geq 0 \implies c = 0$. Note $P(x) = 0$ does work, as the original inequality turns into $2a^2b^2 \geq 0 \geq 0$.

2. $\deg P = 1 \implies P(x) = cx + d$. Substituting this into the original inequality gives

$$2a^2b^2 - 4ab(ca+cb+d) + c(ca+cb+d) + d \geq c(a^2+b^2) + 2d \geq c(a^2 - ab\sqrt{2} + b^2)$$

   This implies $2d \geq c(-ab\sqrt{2})$. However, if $c \neq 0$, then the RHS can attain any value, contradiction. However, this implies $c = 0$, contradicting $\deg P = 1$.

3. $\deg P = 2 \implies P(x) = cx^2 + dx + e \implies$

$$P(P(a)) \geq P(a^2) + P(0) \implies c^3a^4 + O(a^3) \geq ca^4 + O(a^3) \implies c^3 \geq c$$

   and for sufficiently large a we also have $P(a^2) \leq a^4 \implies c \leq 1$. Combining these two gives $c = 1$ or $0 > c \geq -1$. Now we have

$$P(a^2) + P(b^2) \geq P(a^2+b^2) - P(ab\sqrt{2}) \implies$$

$$c(a^4+b^4) + d(a^2+b^2) + 2e \geq c(a^4+b^4+2a^2b^2) + d(a^2+b^2) - c(2a^2b^2) - d(ab\sqrt{2})$$

$$\implies 2e \geq -d(ab\sqrt{2}) \implies d = 0, e \geq 0$$

   Now, we have $P(x) = cx^2 + e$, where $c \in [-1,0) \cup \{1\}$ and $e \geq 0$. Now, we have

$$P(P(0)) \geq 2P(0) \implies ce^2 + e \geq 2e \implies ce^2 \geq e \implies e = 0 \text{ and/or } c > 0 \implies e = 0 \text{ and/or } c = 1$$

   If $c = 1$, then $P(x) = x^2 + e$, so

$$P(P(0)) - 4P(0)a^2 + 2a^4 \geq 2P(a^2) \implies (e^2+e) - 4ea^2 + 2a^4 \geq 2a^4 + 2e$$

However, if $e \neq 0$, then there exists a sufficiently large a which satisfies $LHS < 2a^4$, contradiction. This means that in any scenario, $e = 0$ and $P(x) = cx^2$. Substituting this into the original inequality gives

$$c^3(a+b)^4 + 2a^2b^2 - 4cab(a+b)^2 \geq ca^4 + cb^4 \implies c^3(a+b)^4 + 2a^2b^2 \geq c(a+b)^4 + 2a^2b^2c$$

$$\implies (1-c)((-c^2-c)(a+b)^4 + 2a^2b^2) \geq 0$$

Next, since $c \in [-1,0) \cup \{1\}$, if c is in the negative interval, both factors are positive (as $-c^2 - c \geq 0$), and if $c = 1$, then the LHS is clearly 0. Therefore, all values of $c \in [-1,0) \cup \{1\}$ work.

Now, suppose that $\deg P \geq 3$. If the leading coefficient coefficient ℓ is positive, then there is a sufficiently large a such that

$$2P(a^2) \geq 2\ell a^6 \geq 2a^4 > P(P(0)) - 4P(0)a^2 + 2a^4$$

which is a contradiction. This implies $\ell < 0$. We derived early on that $0 \geq P(a^2) - P(P(a)) + P(0)$. Taking sufficiently large a and sufficiently negative a implies the leading coefficient of the RHS is negative and the leading term has even degree (it is well-known that if $\deg P$ is odd than $P(x)$ can attain infinitely large and infinitely small values).

However, note that the leading term of $P(a^2)$ is $\ell a^{2\deg P}$ and the leading term of $P(P(a))$ is $\ell^{\deg P+1}x^{(\deg P)^2}$. Since $\deg P \geq 3$, then the second leading term, which has degree $(\deg P)^2$, overrides the first leading term, which has degree $2\deg P$. This means that the leading term of the RHS is $-\ell^{\deg P+1}x^{(\deg P)^2}$. This means that $-\ell^{\deg P+1}$ is negative and $(\deg P)^2$ is even. However, since $\ell < 0$, the first part implies that $\deg P + 1$ is even, but $\deg P$ is even by the second part, which is a contradiction. This means that $\deg P$ cannot be greater than 2, and the solutions we have found and verified are indeed the only ones.

Also solved by Fred Frederickson, Utah Valley University, UT, USA.

Olympiad problems

O547. Let a, b, c be the side lengths of a triangle and let R and r be the circumradius and inradius, respectively. Prove that:

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{17r}{18R}\geq\frac{11}{9}.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

Solution by the authors

Let s be the semiperimeter of the triangle ABC. Using $ab+bc+ca=s^2+r^2+4Rr$, we deduce that

$$\sum_{cyc}\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr},\quad \sum_{cyc}\frac{ab}{(a+c)(b+c)}=\frac{s^2+r^2-2Rr}{s^2+r^2+2Rr}.$$

The desired inequality is equivalent to

$$\frac{4(s^2-r^2-Rr)^2}{(s^2+r^2+2Rr)^2}-\frac{2(s^2+r^2-Rr)}{s^2+r^2+2Rr}+\frac{17r}{18R}\geq\frac{11}{9}.$$

Clearing the denominators and expanding, it becomes

$$(14R+17r)s^4-2(116R^2r+96Rr^2-17r^3)s^2+128R^3r^2+124R^2r^3+82Rr^4+17r^5\geq 0,$$

or

$$\left(14+\frac{17r}{R}\right)\frac{s^4}{R^4}-2\left(\frac{116r}{R}+\frac{96r^2}{R^2}-\frac{17r^3}{R^3}\right)\frac{s^2}{R^2}+\frac{128r^2}{R^2}+\frac{124r^3}{R^3}+\frac{82r^4}{R^4}+\frac{17r^5}{R^5}\geq 0.$$

Hence, we need to prove that $f\left(\frac{s^2}{R^2}\right)\geq 0$, where

$$f\left(\frac{s^2}{R^2}\right)=\left(14+\frac{17r}{R}\right)\frac{s^4}{R^4}-2\left(\frac{116r}{R}+\frac{96r^2}{R^2}-\frac{17r^3}{R^3}\right)\frac{s^2}{R^2}+\frac{128r^2}{R^2}+\frac{124r^3}{R^3}+\frac{82r^4}{R^4}+\frac{17r^5}{R^5}.$$

Because

$$s^2\geq 16Rr-5r^2>\frac{116R^2r+96Rr^2-17r^3}{14R+17r},$$

we deduce that f is an increasing function.

If we denote $x^2=1-\frac{2r}{R}\in[0,1)$, then by Blundon Inequality

$$\frac{s^2}{R^2}\geq 2+5(1-x^2)-\frac{(1-x^2)^2}{4}-2x^3=\frac{(1-x)(x+3)^3}{4}.$$

Hence, it suffices to prove that

$$f\left(\frac{(1-x)(x+3)^3}{4}\right)\geq 0,$$

that is

$$\frac{(45-17x^2)(1-x)^2(x+3)^6}{32}-\frac{(1-x)^2(1+x)(x+3)^3(639-158x^2-17x^4)}{16}+\frac{(1-x^2)^2(1701-875x^2+215x^4-17x^6)}{32}\geq 0$$

or after some calculations,

$$4x^2(1-x)^3(x+2)^2(4x+11)\geq 0,$$

which is clearly true. The equality holds when $x=0$, so when the triangle is equilateral. *Also solved by Titu Zvonaru, Comănești, Romania; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Arkady Alt, San Jose, CA, USA; Marie-Nicole Gras, Le Bourg d'Oisans, France.*

O548. Let $m, n, p \geq 2$ be positive integers. Find the number of $n \times p$ matrices with entries in the set $\{1, 2, \ldots, m\}$ such that every element of the matrix is distinct from its row and column neighbors.

Proposed by Mircea Becheanu, Canada

Remark by the author

This problem is a generalization of the problem S539, where $p = 2$. The problem S539 has a very simple solution. We can choose the elements $(a_{11}a_{12})$ of the matrix in $m(m-1)$ ways. Let say, this line is (a, b). The second line should be a pair (x, y) such that $x \neq y$, $x \neq a$ and $y \neq b$. Such a pair is choosen in $m^2 - 3m + 3$ ways. Repeating this for the third line, and so on, we can complete the matrix in $m(m-1)(m^2 - 3m + 3)^{n-1}$ ways.

This method can not be applied in general. The difficulty comes from the fact that after completing the first two rows like in S539, but we can not extend the counting because we do not have information about the number of choices for remaining elements. For example, we can not decide how many ways one can choose a_{33} because we do not know if a_{32} and a_{23} are equal or not.

Ioan Tomescu pointed us that the required number is given by the chromatic polynomial $P(G, m)$ of the grid graph $G(n \times p)$ and this is a NP difficult problem.

O549. Let ABC be a triangle. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq \frac{7}{4}\left(\frac{R}{r}+\frac{r}{R}\right)-\frac{19}{8}\geq \frac{1}{16}\left(21\frac{R}{r}-10\right)\geq \frac{R}{r}.$$

(An improvement of inequality S544.)

Proposed by Marius Stănean, Zalău, România

Solution by Marie-Nicole Gras, Le Bourg d'Oisans, France

We will use the well known relations

$$\cos A=\frac{b^2+c^2-a^2}{2bc},\quad a=2R\sin A,\ \text{and } abc=4srR,\ \text{where } s=\frac{a+b+c}{2}.$$

It follows

$$\begin{aligned}\frac{\cos A}{\sin^2 A}&=\frac{4R^2\cos A}{4R^2\sin^2 A}=\frac{4R^2}{a^2}\,\frac{b^2+c^2-a^2}{2bc}\\&=\frac{b^2+c^2-a^2}{a}\,\frac{R}{2sr}=\left(\frac{b^2+c^2+a^2}{a}-2a\right)\frac{R}{2sr}.\end{aligned}$$

We deduce that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq \frac{7}{4}\left(\frac{R}{r}+\frac{r}{R}\right)-\frac{19}{8}\iff$$

$$F:=\frac{R}{2sr}\left(\frac{(a^2+b^2+c^2)(ab+bc+ca)}{abc}-4s\right)-\frac{7}{4}\left(\frac{R}{r}+\frac{r}{R}\right)+\frac{19}{8}\geq 0. \tag{1}$$

Let $x=s-a$, $y=s-b$ and $z=s-c$ be the Ravi coordinates. We substitute in F

$$a=y+z,\ b=z+x,\ c=x+y,\ \frac{r}{R}=\frac{4xyz}{(y+z)(z+x)(x+y)}.$$

Cleaning denominators and by a straightforward computation, we obtain that $F\geq 0$ is equivalent to $G\geq 0$, with

$$\begin{aligned}G=&\sum_{sym}\left(4x^6y+5x^5y^2-9x^4y^3\right)\\&+\sum_{sym}\left(5x^5yz+15x^4y^2z+9x^3y^3z-29x^3y^2z^2\right).\end{aligned}$$

Applying Muirhead's Inequality gives $G\geq 0$, and we have proved (1).

To conclude, we compute

$$\frac{7}{4}\left(\frac{R}{r}+\frac{r}{R}\right)-\frac{19}{8}-\frac{21}{16}\frac{R}{r}+\frac{5}{8}=\frac{7}{16}\frac{R}{r}+\frac{7}{4}\frac{r}{R}-\frac{7}{4}=\frac{7(R-2r)^2}{16rR}\geq 0$$

and

$$\frac{1}{16}\left(21\frac{R}{r}-10\right)-\frac{R}{r}=\frac{5}{16}\frac{R}{r}-\frac{10}{16}=\frac{5(R-2r)}{16}\geq 0,$$

from Euler's Inequality.

Also solved by Titu Zvonaru, Comănești, Romania; Marin Chirciu, Colegiul Național Zinca Golescu, Pitești, Romania; Arkady Alt, San Jose, CA, USA.

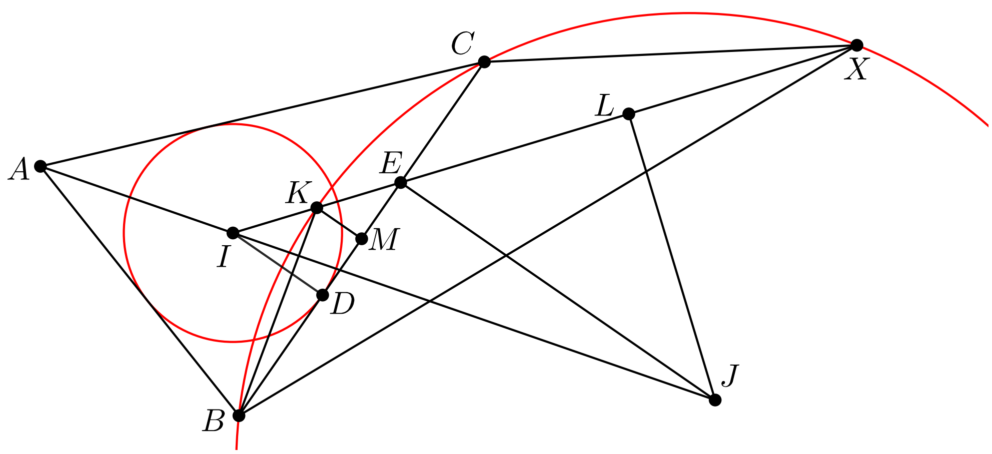
O550. Let ABC be a triangle. Incircle with radius r touches BC at D. Point X lies inside angle BAC and outside triangle and satisfies the following conditions:

$$BD \cdot BX = CD \cdot CX \text{ and } \tan \frac{\angle CXB}{2} = \frac{r}{BC}.$$

Prove that X lies on the A-excircle.

Proposed by Dominik Burek, Krakow, Poland

Solution by Li Zhou, Polk State College, USA



Suppose I and J are the incenter and A-excenter of ABC, respectively. Let M be the midpoint of BC and E be the reflection of D across M. Since $BD = CE$ and $BE = CD$, we have $CE/BE = CX/BX$, thus EX is the bisector of $\angle BXC$ and intersects the circumcircle of BXC at the midpoint K of the arc BC. Since $\angle KBM = \frac{1}{2}\angle CXB$, $KM = r/2$, so K is the midpoint of IE. Let r_A be the A-exradius and L be the orthogonal projection of J on EX. Then $\triangle IDE \sim \triangle ELJ$, so $IE/r = r_A/EL$. Therefore,

$$rr_A = (s-b)(s-c) = CE \cdot BE = KE \cdot EX = \frac{1}{2} IE \cdot EX = rr_A\left(\frac{EX}{2EL}\right),$$

that is, $EL = LX$. Hence, $JX = JE = r_A$, completing the proof.

O551. Let ABC be a triangle and let Δ be its area. Prove that

$$a(s-a)\cos\frac{B-C}{4}+b(s-b)\cos\frac{C-A}{4}+c(s-c)\cos\frac{A-B}{4}\geq 2\sqrt{3}\Delta$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by the author

The inequality to be proved is equivalent to

$$a\frac{\Delta}{s}\cot\frac{A}{2}\cos\frac{B-C}{4}+b\frac{\Delta}{s}\cot\frac{B}{2}\cos\frac{C-A}{4}+c\frac{\Delta}{s}\cot\frac{C}{2}\cos\frac{A-B}{4}\geq 2\sqrt{3}\Delta$$

or

$$a\cot\frac{A}{2}\cos\frac{B-C}{4}+b\cot\frac{B}{2}\cos\frac{C-A}{4}+c\cot\frac{C}{2}\cos\frac{A-B}{4}\geq 2\sqrt{3}s$$

$$\sin A\cot\frac{A}{2}\cos\frac{B-C}{4}+\sin B\cot\frac{B}{2}\cos\frac{C-A}{4}+\sin C\cot\frac{C}{2}\cos\frac{A-B}{4}\geq\sqrt{3}(\sin A+\sin B+\sin C)$$

equivalent to

$$2\cos^2\frac{A}{2}\cos\frac{B-C}{4}+2\cos^2\frac{B}{2}\cos\frac{C-A}{4}+2\cos^2\frac{C}{2}\cos\frac{A-B}{4}\geq\sqrt{3}(\sin A+\sin B+\sin C)$$

Corner transformation: $(A,B,C)\to(\pi-2A,\pi-2B,\pi-2C)$,

$$2\sin^2 A\cos\frac{B-C}{2}+2\sin^2 B\cos\frac{C-A}{2}+2\sin^2 C\cos\frac{A-B}{2}\geq\sqrt{3}(\sin 2A+\sin 2B+\sin 2C)$$

or

$$2\sin A(\sin B+\sin C)\sin\frac{A}{2}+2\sin B(\sin C+\sin A)\sin\frac{B}{2}+2\sin C(\sin A+\sin B)\sin\frac{C}{2}\geq\sqrt{3}(\sin 2A+\sin 2B+\sin 2C)$$

$$\sin A(\sin B+\sin C)\sin\frac{A}{2}+\sin B(\sin C+\sin A)\sin\frac{B}{2}+\sin C(\sin A+\sin B)\sin\frac{C}{2}\geq 2\sqrt{3}\sin A\sin B\sin C \quad (1)$$

Inscribed circle substitution yields $a=y+z, b=z+x, c=x+y (x,y,z\in\mathbb{R}_+)$

$$\sin A=\frac{2\sqrt{xyz(x+y+z)}}{(z+x)(x+y)},\sin B=\frac{2\sqrt{xyz(x+y+z)}}{(x+y)(y+z)},\sin C=\frac{2\sqrt{xyz(x+y+z)}}{(y+z)(z+x)}$$

and

$$\sin\frac{A}{2}=\sqrt{\frac{yz}{(z+x)(x+y)}},\sin\frac{B}{2}=\sqrt{\frac{zx}{(x+y)(y+z)}},\sin\frac{C}{2}=\sqrt{\frac{xy}{(y+z)(z+x)}}$$

Note that (1) can be rewritten as

$$\frac{y^2+z^2+2(xy+yz+zx)}{\sqrt{x(z+x)(x+y)}}+\frac{z^2+x^2+2(xy+yz+zx)}{\sqrt{y(x+y)(y+z)}}+\frac{x^2+y^2+2(xy+yz+zx)}{\sqrt{z(y+z)(z+x)}}\geq 4\sqrt{3(x+y+z)} \qquad (2)$$

Set $x+y+z=1$ and transform inequality (2)

$$\frac{(1-x)^2+2x(1-x)}{\sqrt{x^3+x^2(1-x)+xyz}}+\frac{(1-y)^2+2y(1-y)}{\sqrt{y^3+y^2(1-y)+xyz}}+\frac{(1-z)^2+2z(1-z)}{\sqrt{z^3+z^2(1-z)+xyz}}\geq 4\sqrt{3}.$$

Since $yz\leq\left(\frac{y+z}{2}\right)^2=\frac{1}{4}(1-x)^2$ all we have to prove is

$$\frac{(1-x)^2+2x(1-x)}{\sqrt{x^3+x^2(1-x)+\frac{1}{4}x(1-x)^2}}+\frac{(1-y)^2+2y(1-y)}{\sqrt{y^3+y^2(1-y)+\frac{1}{4}y(1-y)^2}}+\frac{(1-z)^2+2z(1-z)}{\sqrt{z^3+z^2(1-z)+\frac{1}{4}z(1-z)^2}}\geq 4\sqrt{3},$$

or

$$\frac{1-x}{\sqrt{x}}+\frac{1-y}{\sqrt{y}}+\frac{1-z}{\sqrt{z}}\geq 2\sqrt{3}, \tag{3}$$

because $\frac{1-x}{\sqrt{x}}-\frac{4-6x}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{x}}\left(\sqrt{x}-\frac{1}{\sqrt{3}}\right)^2\left(\sqrt{x}+\frac{\sqrt{3}}{2}\right)\geq 0$

Therefore, $\frac{1-x}{\sqrt{x}}\geq\frac{4-6x}{\sqrt{3}}$.

In the same way, two more formulae can be obtained, and it is easy to check that the superposition of the three formulae is valid.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania; Telemachus Baltsavias, Kerameies Junior High School, Kefallonia, Greece; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

O552. Let ABC be a triangle with incenter I. The incircle is tangent to BC, CA, AB at points D, E, F, respectively. Denote by A_1, B_1, C_1 the orthocenters of the triangles AEF, BFD, CDE, respectively.
(1) Prove that circle (DB_1C_1) passes through the foot of the altitude from A of triangle ABC.
(2) Prove that circles $(DB_1C_1), (EC_1A_1), (FA_1B_1)$ have a common point and this point is the Feuerbach point of triangle ABC.

Proposed by Dong Luu, Hanoi National University of Education, Vietnam

Solution by Li Zhou, Polk State College, USA

(1) Let G be the foot of the altitude from A of $\triangle ABC$. Since FD is the perpendicular bisector of B_1I,

$$\frac{BB_1}{BI} = 1 - \frac{B_1I}{BI} = 1 - \frac{B_1I}{FI} \cdot \frac{FI}{BI} = 1 - 2\sin^2\frac{B}{2} = \cos B = \frac{BG}{BA},$$

so $\triangle BGB_1 \sim \triangle BAI$. Therefore, $\angle BGB_1 = \angle BAI = \angle DC_1B_1$, that is, D, B_1, G, C_1 lie on a circle ω_a.

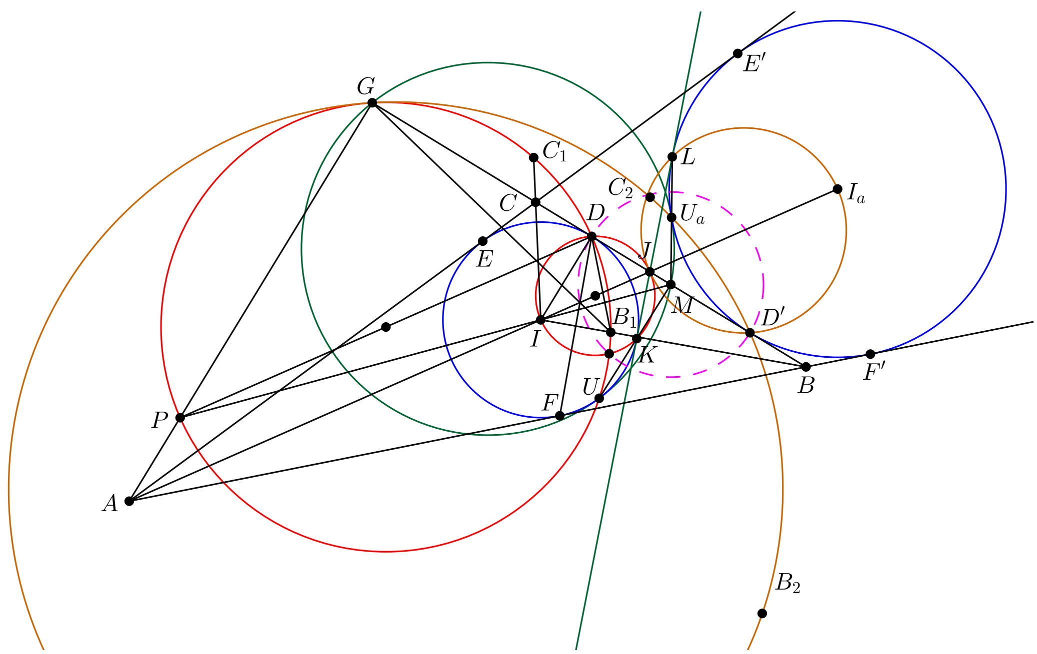
(2) Suppose that ω_a intersects AG at another point P. Then

$$\angle DPG = \angle DB_1G = \angle BDB_1 - \angle BGB_1 = \angle BAG - \angle BAI = \angle IAG,$$

so $PD \parallel AI$, thus $AIDP$ is a parallelogram. Suppose that AI intersects BC at J. Let M be the midpoint of BC and r be the inradius of ABC. We have

$$\frac{PG}{ID} = \frac{AG - r}{r} = \frac{2s}{a} - 1 = \frac{b+c}{a} = \frac{a/2 - b\cos C}{a/2 - (s-c)} = \frac{GM}{DM},$$

so P, I, M are collinear. Hence, the midpoint of PD, the midpoint of IJ, and M are collinear. Now we use the well-known properties of the inversion f centered at M and with radius MD. See T. Andreescu, S. Korsky, & C. Pohoata, *Lemmas in Olympiad Geometry*, XYZ Press, 2016, 218–219. Draw another tangent line from J to the incircle of ABC, with tangency point K. Then $f(JK)$ is the nine-point circle of ABC, and $f(K)$ is the Feuerbach point U of ABC. Since $f(G) = J$ and $f(D) = D$, we see that $f(\omega_a)$ is the circumcircle (DIJ). Therefore, U is on ω_a.



(3) *Comment.* Let I_a be the A-excenter of ABC. The A-excircle of ABC is tangent to BC, CA, AB at points D', E', F', respectively. Denote by B_2 and C_2 the orthocenters of the triangles $BF'D'$ and $CD'E'$, respectively. Then circle $(D'B_2C_2)$ is tangent to ω_a at G and passes through the A-Feuerbach point U_a (where the nine-point circle tangent to the A-excircle of ABC). The proof is very similar.

Also solved by Corneliu Mănescu-Avram, Ploieşti, Romania.